

The boundary value problem for second order ordinary linear differential equations with variable coefficients

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Abstract

In this work two-point boundary value problem for one class of second order ordinary differential equations with variable coefficients is solved.

1 Introduction

Let $0 < x_1 < \infty$, $S[0, x_1]$ is the class of measurable, essentially bounded functions $f(x)$ in $[0, x_1]$ and $W_\infty^2[0, x_1]$ is the class of functions $f(x)$, for which $\frac{d^2 f}{dx^2} \in S[0, x_1]$. The norm of an element from $S[0, x_1]$ is defined by the formulas

$$|f|_0 = \operatorname{ess\,sup}_{x \in [0, x_1]} |f(x)| = \lim_{p \rightarrow \infty} \|f\|_{L_p[0, x_1]}.$$

We consider the equation

$$\frac{d^2 u}{dx^2} + a(x)u = f(x) \quad (1)$$

in interval $[0, x_1]$, where $a(x), f(x) \in S[0, x_1]$.

Two-point boundary value problems for second order ordinary differential equations are classical area of research of the theory of ordinary differential equations and because of their broad application in mechanics, mathematical physics and geometry (see, for example, [1]–[9]) they are still actively investigated. However, in mathematical literature the equations of the form (1) with continuous coefficients are studied and sufficient conditions of resolvability of boundary value problems for them are received. In author's works [10], [11] the general solution of equation (1) is constructed and Cauchy problem for it with initial point $x = 0$ is solved. In this work an explicit form of general solution of equation (1) in class

$$W_\infty^2[0, x_1] \cap C^1[0, x_1], \quad (2)$$

where

$$x_1 < \sqrt{\frac{2}{|a|_0}} \quad (3)$$

is found and next boundary value problem is solved.

Problem D. Find the solution of equation (1) from the class (2) satisfying the conditions

$$u(0) = \alpha, \quad u'(x_1) = \beta, \quad (4)$$

where α, β are given real numbers.

2 Construction of the general solutions to equation (1)

By integrating two times the equation (1), we get

$$u(x) = (Bu)(x) + g(x) + c_1x + c_2, \quad (5)$$

where c_1, c_2 are any real numbers,

$$(Bu)(x) = \int_0^x \int_y^{x_1} a(t)u(t)dt dy, \quad g(x) = \int_0^x \int_y^{x_1} f(t)dt dy.$$

Applying the operator B to both sides of equation (5) we have

$$(Bu)(x) = (B^2u)(x) + (Bg)(x) + c_1a_1(x) + c_2b_1(x), \quad (6)$$

where

$$(B^2u)(x) = (B(Bu)(x))(x),$$

$$a_1(x) = \int_0^x \int_y^{x_1} ta(t)dt dy, \quad b_1(x) = \int_0^x \int_y^{x_1} a(t)dt dy.$$

From (5) and (6) it follows

$$u(x) = (B^2u)(x) + c_1(x + a_1(x)) + c_2(1 + b_1(x)) + g(x) + (Bg)(x). \quad (7)$$

Further we use following functions and operators:

$$a_k(x) = \int_0^x \int_y^{x_1} a(t)a_{k-1}(t)dt dy, \quad b_k(x) = \int_0^x \int_y^{x_1} a(t)b_{k-1}(t)dt dy, \quad (k = 2, 3, \dots),$$

$$(B^k u)(x) = (B(B^{k-1}u)(x))(x), \quad (k = 2, 3, \dots).$$

Applying the operator B to both sides of equation (7) we get

$$(Bu)(x) = (B^3u)(x) + c_1(a_1(x) + a_2(x)) + c_2(b_1(x) + b_2(x)) + (Bg)(x) + (B^2g)(x). \quad (8)$$

From (5) and (8) it follows

$$u(x) = (B^3u)(x) + c_1(x + a_1(x) + a_2(x)) + c_2(1 + b_1(x) + b_2(x)) + g(x) + (Bg)(x) + (B^2g)(x).$$

Continuing this procedure n times we obtain the following integral representation of solutions of equation (1):

$$u(x) = (B^n u)(x) + c_1\left(x + \sum_{k=1}^{n-1} a_k(x)\right) + c_2\left(1 + \sum_{k=1}^{n-1} b_k(x)\right) + \sum_{k=0}^{n-1} (B^k g)(x), \quad (9)$$

where $(B^0 g)(x) = g(x)$.

Let $u(x) \in C[0, x_1]$. The following inequalities are easily obtained:

$$|(B^n u)(x)| \leq 2|u|_1 \cdot \frac{|a|_0^n \cdot x_1^{2n}}{2^n}, \quad (n = 1, 2, \dots), \quad (10)$$

$$|a_k(x)| < \frac{|a|_0^k \cdot x_1^{2k}}{2^k} \cdot x, \quad |b_k(x)| < \frac{2 \cdot |a|_0^k \cdot x_1^{2k}}{2^k}, \quad (k = 1, 2, \dots), \quad (11)$$

where $|f|_1 = \max_{x \in [0, x_1]} |f(x)|$.

Passing to the limit with $n \rightarrow \infty$ in the representation (9) and taking inequalities (10), (3) into account we get

$$u(x) = c_1 I_1(x) + c_2 I_2(x) + F(x), \quad (12)$$

where

$$I_1(x) = x + \sum_{k=1}^{\infty} a_k(x), \quad I_2(x) = 1 + \sum_{k=1}^{\infty} b_k(x), \quad F(x) = \sum_{k=0}^{\infty} (B^k g)(x).$$

Using the inequalities (10), (11), we receive

$$|I_1(x)| \leq \frac{2}{2 - |a|_0 \cdot x_1^2}, \quad |I_2(x)| < \frac{2 + |a|_0 \cdot x_1^2}{2 - |a|_0 \cdot x_1^2}, \quad |F(x)| \leq |g|_1 \frac{2 + |a|_0 \cdot x_1^2}{2 - |a|_0 \cdot x_1^2}. \quad (13)$$

From the form of functions $I_1(x), I_2(x)$ and $F(x)$ for $x \in [0, x_1]$, where x_1 satisfies inequality (3), it follows

$$I_1'(x) = 1 + \int_x^{x_1} a(t) I_1(t) dt, \quad I_2'(x) = \int_x^{x_1} a(t) I_2(t) dt, \quad (14)$$

$$F'(x) = - \int_x^{x_1} f(t) dt + \int_x^{x_1} a(t) F(t) dt,$$

$$I_1''(x) = -a(x) I_1(x), \quad I_2''(x) = -a(x) I_2(x), \quad (15)$$

$$F''(x) = f(x) - a(x) F(x). \quad (16)$$

Considering (14) and the form of functions $I_1(x), I_2(x), F(x)$ we get

$$I_1(0) = F(0) = F'(x_1) = I_2'(x_1) = 0, \quad I_2(0) = I_1'(x_1) = 1. \quad (17)$$

From (15) and (16) it follows, that functions $I_1(x), I_2(x)$ are particular solutions from class (2) of homogeneous equations

$$\frac{d^2 u}{dx^2} + a(x) u = 0,$$

and the function $F(x)$ is solution of non-homogeneous equation (1).

From (15) and (17) we see that the Wronskian $W(x)$ is equal to $-I_2(x_1)$. Therefore if $I_2(x_1) \neq 0$ then the functions $I_1(x)$ and $I_2(x)$ are linear independent on $[0, x_1]$ and the general solution to equation (1) is determined by the formula (12).

Hence, we proved the following theorem.

Theorem 2.1 *If $I_2(x_1) \neq 0$ then the function $u(x)$, given by the formula (12), is a general solution of equation (1) from class (2).*

3 Solution of boundary value problem

For equation (1) we consider the problem D. To solve the problem D we use the solution of equation (1), given by the formula (12). Substituting the function $u(x)$, given by formula (12), into boundary conditions (4) and taking into account (17) we have

$$c_2 = \alpha, \quad c_1 = \beta.$$

Hence, the solution of problem D has a form

$$u(x) = \beta I_1(x) + \alpha I_2(x) + F(x). \quad (18)$$

Therefore, the following theorem is proved.

Theorem 3.1 *If $I_2(x_1) \neq 0$ then the problem D has solution, which is given by the formula (18).*

Remark 3.1 *Obviously, that the results of present work and take place for $a(x), f(x) \in C[0, x_1]$. In this case the solutions given by the formulas (12) and (18) belong to the class $C^2[0, x_1]$.*

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